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SURFACES OF WEAK DISCONTINUITY FOR CUBICALLY ANISOTROPIC SOLID BODIES

S. M. Bosyakov[a] and O. N. Sklyar[b]

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In the context of the general theory of characteristics, the velocities of propagation of the discontinuity surfaces in cubically anisotropic deformable media are obtained.

We consider solid bodies, the elastic properties of which are described by the Hooke law [1, 2]

$$\sigma_{ii} = (A_{11} - A_{12})\, e_{ii} + A_{12}\theta\,,\ \sigma_{ij} = 2A_{44}e_{ij}\,,\ i \neq j\,,$$

$$e_{ij} = \frac{1}{2}\,(\partial_i u_j + \partial_j u_i)\,,\ \ \theta = \sum_{k=1}^{3} \partial_k u_k\,, \tag{1}$$

$$A_{ij} = \text{const}\,,\ \ \partial_i = \frac{\partial}{\partial x_i}\,,\ \ i, j = \overline{1, 3}\,.$$

Here $\vec{u} = (u_1, u_2, u_3)$.

We write the equations of motion in the following form [1]:

$$\sum_{j=1}^{3} \partial_j \sigma_{ij} + X_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}\,. \tag{2}$$

Then, substituting the expressions for the stress tensor from (1) into (2), we arrive at

$$(\Delta + \varepsilon \partial_i^2)\, u_i + \sigma \partial_i \theta + \overline{X}_i = \rho \frac{\partial^2 u_i}{\partial t^2}\,, \tag{3}$$

where

$$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2\,,\ \ \varepsilon = \frac{A_{11} - A_{12}}{A_{44}} - 2\,,\ \ \sigma = \frac{A_{12}}{A_{44}} + 1\,,\ \ \rho = \frac{\rho_0}{A_{44}}\,,\ \ \overline{X}_i = \frac{X_i}{A_{44}}\,.$$

We will consider the Cauchy problem for the system of equations (3) with the following initial conditions:

$$u_i\big|_{t=0} = f_i\,(0, x_1, x_2, x_3)\,,\ \ \frac{\partial u_i}{\partial t}\bigg|_{t=0} = \varphi_i\,(0, x_1, x_2, x_3)\,. \tag{4}$$

In the general case, we will prescribe the proper functions u_i and their derivatives of the first order on the surface $Z = Z(t, x_1, x_2, x_3)$ and find an equation of characteristics from the condition of impossibility of determination of $\partial^2 u_i/\partial t^2$ [3].

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[a]Belarusian State University; [b]Belarusian State Polytechnic Academy, Minsk, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 3, pp. 662-664, May–June, 2000. Original article submitted August 3, 1999.

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We introduce new variables into (3) in accordance with the scheme [4]

$$\begin{pmatrix} t \equiv x_0, x_1, x_2, x_3 \\ Z \equiv Z_0, Z_1, Z_2, Z_3 \end{pmatrix}.$$

We write the expressions of the derivatives with respect to the former variables in terms of the derivatives with respect to the new variables:

$$\frac{\partial u_i}{\partial x_k} = \sum_{n=0}^{3} \frac{\partial u_i}{\partial Z_n} \frac{\partial Z_n}{\partial x_k},$$

$$\frac{\partial^2 u_i}{\partial x_l \partial x_k} = \sum_{m,n=0}^{3} \frac{\partial^2 u_i}{\partial Z_m \partial Z_n} \frac{\partial Z_m}{\partial x_l} \frac{\partial Z_n}{\partial x_k} + \sum_{n=0}^{3} \frac{\partial u_i}{\partial Z_n} \frac{\partial^2 Z_n}{\partial x_l \partial x_k}. \tag{5}$$

Substituting (5) into (3) and writing only those terms that contain $\partial^2 u_i/\partial Z^2$, we arrive at the following system of equations in new variables:

$$\frac{\partial^2 u_i}{\partial Z^2} g^2 + \varepsilon \frac{\partial^2 u_i}{\partial Z^2} p_i^2 + \sigma p_i \sum_{k=1}^{3} p_k \frac{\partial^2 u_k}{\partial Z^2} - \rho \frac{\partial^2 u_i}{\partial z^2} p_0^2 + \ldots = 0. \tag{6}$$

Here

$$p_i = \frac{\partial Z}{\partial x_i}, \quad g^2 = \sum_{k=1}^{3} p_k^2, \quad p_0 = \frac{\partial Z}{\partial t}.$$

Let ω_{ij} be the coefficients at the second derivative function u_i with respect to Z^2 in the j th equation of system (6), $i, j = \overline{1, 3}$. In our case

$$\omega_{ij} = \sigma p_i p_j + \delta_{ij} (g^2 + \varepsilon p_i^2 - \rho p_0^2), \quad i, j = \overline{1, 3}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{7}$$

The surface $Z = 0$ will be the characteristic one if the determinant composed of the coefficients at $\partial^2 u_i/\partial Z^2$ is equal to zero, i.e., if the equality $\det \|\omega_{ij}\| = 0$ is fulfilled.

Expanding the determinant, we obtain the following equation of propagation of the characteristics:

$$(g^2 - \rho p_0^2)^3 + (\varepsilon + \sigma) g^2 (g^2 - \rho p_0^2)^2 + (\varepsilon^2 + 2\varepsilon\sigma)(g^2 - \rho p_0^2)(p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2) +$$
$$+ (2\sigma^3 - 3\sigma^2 (\varepsilon + \sigma) + (\varepsilon + \sigma)^3) p_1^2 p_2^2 p_3^2 = 0. \tag{8}$$

Taking into consideration that the velocity of displacement of the characteristic surface is $P = -p_0/g$ and the direction cosines of the normal are $\cos \alpha_k = p_k/g$, we divide (8) by g^6. We will write the obtained equation under the assumption that the direction of propagation of the characteristic surface coincides with the direction of the normal to this surface:

$$(1 - \rho P^2)^3 + (\varepsilon + \sigma)(1 - \rho P^2)^2 = 0.$$

Whence we have the following possible velocities of displacement of the discontinuity surface:

$$P_1 = \sqrt{\left(\frac{1}{\rho}\right)}, \quad P_2 = \sqrt{\left(\frac{1 + \varepsilon + \sigma}{\rho}\right)}.$$

Now we clear up the character of discontinuity. We assume that the functions u_i as such and their derivatives of the first order remain continuous on passing across the surface $Z = 0$, while the derivatives of the second order $\partial^2 u_i/\partial x_k \partial x_l$, $k, l = \overline{0, 3}$, undergo discontinuity. Under kinematic conditions of compatibility [3], each such derivative experiences an abrupt change that is proportional to the components of the normal of the characteristic surface. Whence follows the existence of proportionality coefficients h_k such that

$$[u_{x_k} u_{x_l}] = \overset{+}{u}_{x_k x_l} - \overset{-}{u}_{x_k x_l} = h_k Z_{x_l} .$$

Substituting these expressions into the kinematic conditions of compatibility, we obtain the system of homogeneous equations for the multipliers h_j, the determinant of which is equal to zero:

$$\sum_{j=1}^{3} \omega_{ij} h_j = 0 . \tag{9}$$

By virtue of (7) Eqs. (9) acquire the form

$$(g^2 + \varepsilon p_i^2 - \rho p_0^2)\, h_i + \sigma p_i \sum_{k=1}^{3} p_k h_k = 0 , \quad i = \overline{1, 3} .$$

Since $p_i = g \cos(\vec{n}, x_i)$, where $\vec{n}$ is the direction of the normal to the surface $Z = 0$, we can represent the above equations as

$$(g^2 + \varepsilon g^2 \cos^2(n, x_i) - \rho p_0^2)\, h_i + \sigma g^2 \cos(n, x_i) \sum_{k=1}^{3} \cos(n, x_k)\, h_k = 0$$

or

$$(g^2 - \rho p_0^2)\, h_i + (\varepsilon + \sigma)\, g^2 \cos(n, x_i)\, h_n = 0 ,$$

where h_n is the projection of the vector $\vec{h} = (h_1, h_2, h_3)$ onto the normal $\vec{n}$ to the surface $Z = 0$. In vector form,

$$(g^2 - \rho p_0^2)\, \vec{h} + (\varepsilon + \sigma)\, g^2 h_n \vec{e} = 0 .$$

If we take the displacement velocity P_1, then the coefficient on $\vec{h}$ is equal to zero and $h_n = 0$, i.e., $\vec{h}$ lies in the plane tangential to the surface $Z = 0$; therefore the discontinuity surface is a transverse wave. If we take the velocity P_2, then $\vec{h}$ differs from $\vec{e}$ in a numerical multiplier, i.e., $\vec{h}$ is directed along the normal to the surface; consequently, $Z = 0$ is a longitudinal wave.

Thus, a solution of system (3) exists which as such is continuous together with the derivatives of the first order, and its second derivatives have a discontinuity of the first kind on the surface $Z = 0$. This means that the solution of Eqs. (3) has a weak discontinuity on the surface $Z = 0$ and only the characteristic surface can be the surface of weak discontinuity.

This effect was known earlier for the case of isotropic bodies [3].

NOTATION

e_{ij} and σ_{ij}, components of the strain tensor and the stress tensor; A_{11}, A_{12}, and A_{44}, material constants of the elastic medium; $\vec{u}$, translational vector of the body points; X_i, mass forces; ρ_0, density of the elastic medium; $\Delta \sum_{k=1}^{3} \partial_k^2$, $p_k = \frac{\partial Z}{\partial x_k}$, and $\theta = \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_k}$, volume strain; cos α_i, direction cosines of the normal; $\vec{e}$, unit vector of the normal to the surface $Z = 0$, $i, j = \overline{1, 3}$.

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